



Early Journal Content on JSTOR, Free to Anyone in the World

This article is one of nearly 500,000 scholarly works digitized and made freely available to everyone in the world by JSTOR.

Known as the Early Journal Content, this set of works include research articles, news, letters, and other writings published in more than 200 of the oldest leading academic journals. The works date from the mid-seventeenth to the early twentieth centuries.

We encourage people to read and share the Early Journal Content openly and to tell others that this resource exists. People may post this content online or redistribute in any way for non-commercial purposes.

Read more about Early Journal Content at <http://about.jstor.org/participate-jstor/individuals/early-journal-content>.

JSTOR is a digital library of academic journals, books, and primary source objects. JSTOR helps people discover, use, and build upon a wide range of content through a powerful research and teaching platform, and preserves this content for future generations. JSTOR is part of ITHAKA, a not-for-profit organization that also includes Ithaka S+R and Portico. For more information about JSTOR, please contact support@jstor.org.

III. CONCYCLIC POINTS ON AN EQUILATERAL HYPERBOLA AND ON ITS INVERSES.

By R. M. MATHEWS, Wesleyan University.

The equation of an equilateral hyperbola may be written

$$xy - x - y = 0, \quad (1)$$

where the origin is a vertex and the axes of coördinates are parallel to the asymptotes. Writing $y = mx$, we can obtain the parametric equations

$$x = \frac{1+m}{m}, \quad y = 1+m. \quad (2)$$

There is one to one correspondence between all finite values of m (except zero) and all finite points on the curve.

These expressions (2) substituted in the equation of an arbitrary circle

$$x^2 + y^2 + 2gx + 2fy + c = 0, \quad (3)$$

lead to the quartic equation

$$m^4 + 2(1+f)m^3 + (2+2g+2f+c)m^2 + 2(1+g)m + 1 = 0. \quad (4)$$

The roots of this equation are the parameters of the points of intersection of the circle with the hyperbola. It is easy to show that the necessary and sufficient condition that four points of an equilateral hyperbola (2) be concyclic is that their parameters be such that

$$m_1 m_2 m_3 m_4 = 1.$$

Take three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ of parameters k_i , l_i , m_i , respectively ($i = 1, 2, 3, 4$). The circle through A_i , B_i , C_i will cut the hyperbola in a fourth point D_i (parameter n_i) such that

$$k_i l_i m_i n_i = 1 \quad (i = 1, 2, 3, 4).$$

From these equations, with the help of the facts that the sets of A 's, B 's and C 's are severally concyclic, we obtain

$$n_1 n_2 n_3 n_4 = 1.$$

Thus we have proved the theorem:

Given three sets of four concyclic points $\{A_i\}$, $\{B_i\}$, $\{C_i\}$ ($i = 1, 2, 3, 4$) on an equilateral hyperbola; let the circle through $A_i B_i C_i$ cut the curve in a fourth point D_i ; then the four points D_i are concyclic.

As specializations of this proposition we have two known theorems, the first due to Cazamian.¹

Two circles meet an equilateral hyperbola in $\{A_i\}$ and $\{C_i\}$ ($i = 1, 2, 3, 4$),

¹ "Sur l'hyperbole équilatère et sur ses inverses," *Nouvelles Annales de Mathématiques*, series 3, vol. 13, 1894, pp. 265-280.

respectively; draw the circle through A_i and tangent to the curve at C_i ; the four circles so obtained meet the conic again in four concyclic points.

The four circles which osculate an equilateral hyperbola in four concyclic points cut the curve again in four concyclic points.

When an equilateral hyperbola is inverted about a point not on it, the transform is a bicircular quartic with a singular point at the center of inversion and the tangents there are at right angles (for they are parallel to the asymptotes of the conic). Thus the quartic is *orthotomic*. In particular, when the point of inversion is at the center of the hyperbola, the transform is a lemniscate. If the hyperbola be inverted about a point on it, the transform is a strophoid, oblique, in general, but a right strophoid, or Booth's logocyclica, when the center of inversion is a vertex of the conic. Every orthotomic bicircular quartic and every strophoid may be obtained by such an inversion.

As an inversion carries circles into circles, in general, the theorems just stated hold true when "equilateral hyperbola" is replaced by "orthotomic bicircular quartic" or by "strophoid," and the circles are conditioned as not passing through the node.

When one of the circles of the hyperbola passes through the center of inversion, special theorems result after the inversion. For example, in the main theorem for a strophoid let A_1 be at ∞ , and then:

Let A_2, A_3, A_4 be three collinear points of a strophoid while $\{B_i\}$ and $\{C_i\}$ ($i = 1, 2, 3, 4$) are two sets of concyclic points. Let the line B_1C_1 cut the curve in D_1 , while the circle on $A_iB_iC_i$ ($i = 2, 3, 4$) meets the cubic in D_i . Then the four points $\{D_i\}$ are concyclic.

In the case of one of the quartics the initial points may be arranged as: (1) one set of four collinear points with two sets of concyclic; (2) two sets of collinear points with one concyclic; (3) three sets of collinear points. None of the lines or circles of the quartic may pass through the singular point.

It is evident that the main theorem carries on to curves of still higher degree obtained by successive inversions.

On returning to the original theorem, we observe that there is no order assigned for the choice of a point triple to determine an (ABC) circle. When the three sets of concyclic points are assigned, it is evident that there are 64 circles (ABC) possible; each is determined by a triple of points, one from each set. Accordingly, there are 64 D -points which are concyclic in groups of four. To find how many of the possible ${}_4C_4$ are admissible here we observe: first, that each D -set is determined by a certain four of (ABC) circles; second, that in this set must be one circle for each of the A 's; third, under this line of A 's must be written some permutation of the four B 's and some permutation of the four C 's. Thus there are $4 \cdot 4 = 576$ such sets and so 576 D -circles.

There are now $12 + 64 = 76$ points concyclic by fours in at least $576 + 64 + 3 = 643$ ways. It is easy to show that if a set of D -points be used with the B 's and C 's, the original A -set will be reached only once. It remains, then, to determine the conditions under which the configuration is closed as it stands, or will be closed after the addition of other points.